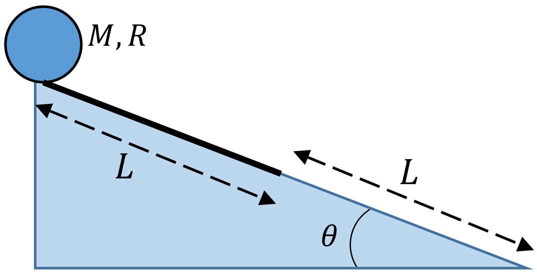
PHYS 101 – General Physics I Final Exam

Duration: 120 minutes Tuesday, 27 December 2022; 09:30

1 . An inclined plane with angle of inclination θ has two sections of length L; the top half has friction while the bottom half is frictionless as shown in the figure. A disc of radius R and mass M ($I_{CM} = MR^2/2$) is released from the top. The friction in the upper part is strong enough so that the disc rolls without slipping in that part. Gravitational acceleration is g.



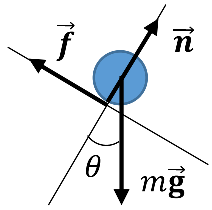
(a) (9 Pts.) What is its center of mass speed when the disc reaches the bottom of the inclined plane?

(b) (8 Pts.) What is the angular velocity of the disc when it reaches the bottom of the inclined plane?

(c) (8 Pts.) How long does it take for the disc to reach the bottom?

Solution:

(a) From the free body diagram for the disc at the top half, we have



$$Mg\sin\theta - f = Ma_{1c}\,, \qquad Rf = \frac{1}{2}MR^2\alpha\,, \qquad \alpha = \frac{a_{1c}}{R} \;\rightarrow\; f = \frac{1}{2}Ma_{1c} \;\rightarrow\; a_{1c} = \frac{2}{3}\mathrm{g}\sin\theta.$$

Speed of the center of the disk v_1 after the covering the distance L is found by using

$$v_1^2 - v_0^2 = 2a_{1c}L \quad\rightarrow\quad v_1^2 = \frac{4}{3}\mathrm{g}L\sin\theta\,.$$

No friction at the bottom half of the inclined plane means in that part we have $a_{2c} = \mathrm{g}\sin\theta$. Speed of the center of the disc v_2 at the bottom of the inclined plane is found as

$$v_2^2 - v_1^2 = 2a_{2c}L \quad\rightarrow\quad v_2^2 = \frac{4}{3}\mathrm{g}L\sin\theta + 2\mathrm{g}L\sin\theta \quad\rightarrow\quad v_2 = \sqrt{\frac{10}{3}gL\sin\theta}\,.$$

(b) Since there is friction only at the first half of the inclined plane, angular speed of the disk will not change in the second half.

$$\omega_2 = \omega_1 = \frac{v_1}{R} \rightarrow \quad \omega_2 = \frac{1}{R}\sqrt{\frac{4}{3}\mathrm{g}L\sin\theta}\,.$$

(c) For the top half

$$L = \frac{1}{2}a_{1c}t_1^2 \quad\rightarrow\quad t_1 = \sqrt{\frac{3L}{\mathrm{g}\sin\theta}} = 3\sqrt{\frac{L}{3\mathrm{g}\sin\theta}}.$$

For the bottom half

$$L = v_1t_2 + \frac{1}{2}a_{2c}t_2^2 \quad\rightarrow\quad t_2 = \sqrt{\frac{10L}{3\mathrm{g}\sin\theta}} - \sqrt{\frac{4L}{3\mathrm{g}\sin\theta}} = \left(\sqrt{10} - 2\right)\sqrt{\frac{L}{3\mathrm{g}\sin\theta}}.$$

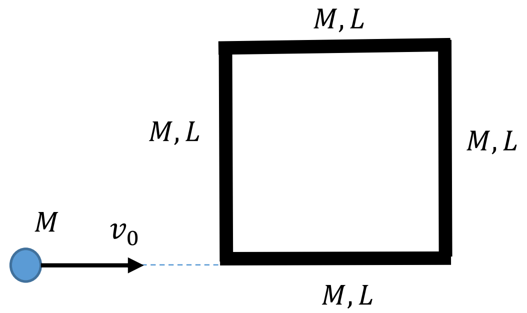
Total time taken is

$$t_T = t_1 + t_2 = \left(\sqrt{10} + 1\right)\sqrt{\frac{L}{3\mathrm{g}\sin\theta}}.$$

2. A rigid square is made by joining 4 homogenous rods of length L and mass M as shown in the figure. For a single rod the moment of inertia about its center of mass is $I_{CM} = ML^2/12$.

(a) (8 Pts.) What is the moment of inertia of the square around an axis perpendicular to the plane of the page, and passing through its center of mass? (Call this quantity I_0 and continue the problem even if you cannot solve this part.)

Solution: $I_0 = 4\left(\frac{ML^2}{12} + \frac{ML^2}{4}\right) = \frac{4}{3}ML^2$.



The square is resting on a horizontal frictionless plane, a point of object of mass M (same as one of the rods forming the square) hits its corner with a velocity v_0 parallel to the bottom side of the square as shown in the figure. The collision is completely inelastic, the mass sticks to the corner of the square and objects move together after the collision

(b) (4 Pts.) Which quantities are conserved in the collision (Correct choice +1, Wrong choice -1 for each)

	Conserved	Not Conserved
Mechanical Energy		X
Linear Momentum	X	
Angular Momentum with respect to the collision point	X	
Angular Momentum with respect to the center of mass of the system	X	

(c) (5 Pts.) What is the speed of the center of mass of the system after the collision?

(d) (8 Pts.) What is the angular velocity of the system around its center of mass after the collision?

Solution:

(c) Linear momentum is conserved in the collision means the motion will be in the horizontal direction.

$$Mv_0 = 5Mv_{CM} \quad \rightarrow \quad v_{CM} = \frac{v_0}{5}.$$

(d) The system will be revolving around its center of mass which, taking the collision point as the origin, will be at $(2L/5\,, 2L/5)$. Distance between this point and the center of the square is $d = \sqrt{2}L/10$. Moment of inertia of the system around the new center of mass is

$$I_{CM} = I_0 + 4M\left(\sqrt{2}L/10\right)^2 + M\left(2\sqrt{2}L/5\right)^2 = I_0 + \frac{2}{5}ML^2 \quad \rightarrow \quad I_{CM} = \frac{26}{15}ML^2 .$$

Conservation of angular momentum with respect to this point gives

$$Mv_0\left(\frac{2L}{5}\right) = \left(\frac{26}{15}ML^2\right)\omega \quad \rightarrow \quad \omega = \frac{3v_0}{13L}.$$

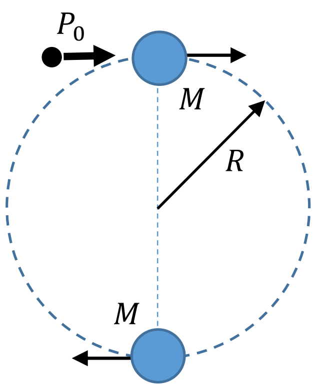
3. On September 26, 2022, NASA's DART mission deflected the motion of an asteroid system by an impacting spacecraft. Answer the following questions for a simplified version of the event:

(a) (5 Pts.) The asteroid system hit by the spacecraft is formed by two almost spherical rocks (called Didymos and Dimorphos), rotating around their common center of mass. Assume that both rocks are spherical, have equal mass M, and they have a stable circular orbit around their common center of mass so that the distance between their centers is $2R$. What is the period of their motion?

Solution: (a)

$$\frac{GM^2}{4R^2} = M\frac{v^2}{R} \quad \rightarrow \quad v = \sqrt{\frac{GM}{4R}}\,, \qquad T = \frac{2\pi R}{v} \quad \rightarrow \quad T = 4\pi\sqrt{\frac{R^3}{GM}}\,.$$

The impacting spacecraft hits one of the rocks in a direction perpendicular to the line connecting the two rocks as shown in the figure. The mass of the spacecraft is a million times smaller than that of the rocks, so its gravitational interaction or added mass can be neglected. But it is moving with an extremely high velocity so its momentum just before the impact P_0 cannot be neglected. The collision of the spacecraft is completely inelastic. (Express all your results in terms of M, R, P_0 and Newton's constant G.)



(b) (4 Pts.) Find the magnitude of the velocity of one of the rocks with respect to the other after the collision.

(c) (4 Pts.) Find the speed of the common center of mass after the collision.

(d) (4 Pts.) What is the angular momentum with respect to the common center of mass of the two rock system after the collision? (You can treat the asteroids as point particles.)

(e) (8 Pts.) What is the minimum P_0 so that the two astreoids can get infinitely far away from each other?

Solution: (b) Linear momentum of the system of spacecraft and the rock it collides with is conserved in the collision because the external force (gravitational attraction of the second rock) is in a perpendicular direction. This gives

$$P_0 + Mv = Mv' \quad \rightarrow \quad v' = v + \frac{P_0}{M} \quad \rightarrow \quad v_{\text{rel}} = v + v' \quad \rightarrow \quad v_{\text{rel}} = 2\sqrt{\frac{GM}{4R}} + \frac{P_0}{M}\,.$$

(c) Linear momentum of the whole system (two rocks and the spacecraft) is also conserved because there are no external forces. This gives

$$P_0 = 2M\,V_{CM} \quad \rightarrow \quad V_{CM} = \frac{P_0}{2M}\,.$$

Note that one gets the same result by writing $V_{CM} = (Mv' - Mv)/2M$.

(d) Since the added mass is neglected, position of the center of mass does not change during the collision. Also, angular momentum is conserved during the collision. Therefore,

$$L_f = L_i = RP_0 + I\omega = RP_0 + 2MRv \quad \rightarrow \quad L_f = RP_0 + \sqrt{GM^3R}\,.$$

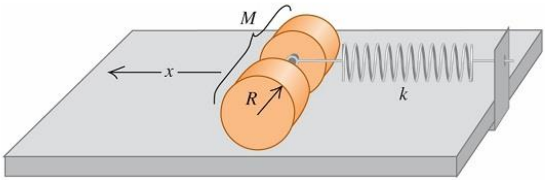
(e) The two asteroids can get infinitely far away from each other if the total mechanical energy with respect to their center of mass is at least zero. Speed of each asteroid with respect to the center of mass immediately after the collision is $v + P_0/2M$. Therefore,

$$E = 2\left[\frac{1}{2}M\left(v + \frac{P_0}{2M}\right)^2\right] - \frac{GM^2}{2R} \geq 0 \quad \rightarrow \quad P_0 \geq \left(\sqrt{2} - 1\right)\sqrt{\frac{GM^3}{R}}\,.$$

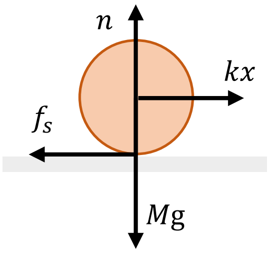
4. Two uniform, solid cylinders of radius R and total mass M are connected along their common axis by a short, rod of negligible mass and rest on a horizontal table top, as shown in the figure. A frictionless ring at the center of the rod is attached to a spring with force constant k while the other end of the spring is fixed. The cylinders are pulled to the left, stretching the spring, and then released from rest. Assume that the friction between the surface and the cylinders is large enough to ensure that they roll without slipping on the surface. Gravitational acceleration g, and $I_{CM} = MR^2/2$ for the cylinders.

(a) (15 Pts.) Find the frequency of oscillations of the system.

(b) (10 Pts.) What is the minimum coefficient of static friction $\mu_{s\,\min}$ between the surface and the cylinders that can ensure that no slipping happens for an oscillation with total mechanical energy E?



Solution: Method 1: Rolling means $a_c = -R\alpha$. Newton's second law is written as



$$f_s - kx = Ma_c\,, \qquad Rf_s = I\alpha = \frac{a_c}{R} \;\rightarrow\; f_s = -\frac{1}{2}Ma_c \;\rightarrow\; \frac{3}{2}Ma_c = -kx\,.$$

$$a_c = \frac{d^2x}{dt^2} \;\rightarrow\; \frac{d^2x}{dt^2} = -\left(\frac{2k}{3M}\right)x \;\rightarrow\; \omega = \sqrt{\frac{2k}{3M}}\,.$$

Method 2: Writing the total energy of the cylinders and using the rolling condition $v = R\,\omega$, we have

$$E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 = \frac{1}{2}\left(M + \frac{I}{R^2}\right)v^2 + \frac{1}{2}kx^2 \;\rightarrow\; E = \frac{3}{4}Mv^2 + \frac{1}{2}kx^2\,.$$

Since the total energy is constant during the motion,

$$\frac{dE}{dt} = \frac{3}{2}Mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0 \;\rightarrow\; \left(\frac{3}{2}M\frac{d^2x}{dt^2} + kx\right)v = 0 \;\rightarrow\; \frac{d^2x}{dt^2} = -\left(\frac{2k}{3M}\right)x\,.$$

(b) Writing Newton's second law in the vertical direction, we have

$$n - Mg = 0 \;\rightarrow\; n = Mg \;\rightarrow\; f_s \le \mu_s Mg\,.$$

Since

$$|f_s| = Ma_c/2 \;\rightarrow\; a_c \le 2\mu_{s\,\min}g\,, \qquad a_{\max} = 2\mu_{s\,\min}g\,.$$

During the oscillatory motion $E = kA^2/2$, and is constant. Also, the maximum acceleration is

$$a_{\max} = A\omega^2 = \frac{2\sqrt{2kE}}{3M} \;\rightarrow\; 2\mu_{s\,\min}g = \frac{2\sqrt{2kE}}{3M} \;\rightarrow\; \mu_{s\,\min} = \frac{\sqrt{2kE}}{3Mg}\,.$$